MATHEMATICAL INDUCTION

Example

$1 = 1$

$1+3 = 4$

$1+3+5 = 9$

$1+3+5+7 = 16$

$1+3+5+7+9 = 25$

$\cdots\cdots$

Wow! It looks like

$$\boxed{1+3+5+\cdots+(2n-1) = n^2}$$

Is this true ?

This is a statement $\varphi(n)$ about the number $n \in \mathbb{N}$ we want to determine whether the $\varphi(n)$ holds for <u>all</u> $n \in \mathbb{N}$. The example shows $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, $\varphi(4)$ and $\varphi(5)$ are all true.

Theorem

$$1+3+5+\cdots+(2n-1) = n^2$$

holds for all $n \in \mathbb{N}$

Proof By mathematical induction

Let $\varphi(n)$ be the statement
"$1+3+5+\cdots+(2n-1) = n^2$".

STEP ① "BASE CASE"

The statement $\varphi(1)$ is
$$1 = 1^2$$
which is TRUE

STEP ② "INDUCTION STEP"

We now assume $n \in \mathbb{N}$ is an arbitrary number and we assume that $\varphi(n)$ is TRUE

INDUCTION HYPOTHESIS : $\varphi(n)$

So $\quad 1+3+5+\cdots+(2n-1) = n^2$

So $\quad 1+3+5+\cdots+(2n-1)+(2n+1)$
$\qquad = n^2+2n+1$

So $\quad 1+3+5+\cdots+(2n-1)+(2(n+1)-1)$
$\qquad = (n+1)^2$

$$1 + 3 + \dots + (2(n+1)-1) = (n+1)^2$$

THIS IS JUST THE STATEMENT $\varphi(n+1)$

STEP ③ PUTTING IT ALL TOGETHER

Step ① shows that $\varphi(1)$ is true

Step ② shows that

$$\text{for all } n \in \mathbb{N} \ (\varphi(n) \Rightarrow \varphi(n+1)) \quad ⊛$$

So by $\varphi(1)$ and ⊛ we deduce $\varphi(2)$
by $\varphi(2)$ and ⊛ we deduce $\varphi(3)$
by $\varphi(3)$ and ⊛ we deduce $\varphi(4)$
- - -
and so on.

Applying ⊛ k times to $\varphi(1)$ we may deduce $\varphi(k+1)$. We can do this for any $k \in \mathbb{N}$ we like.

So:

$\varphi(n)$ is true for all $n \in \mathbb{N}$ by Mathematical induction.

(Step ① & Step ② are essential. You normally only write this instead of step ③.)

The <u>induction step</u> is usually the hardest one. The <u>base case</u> is, however, essential!

<u>INCORRECT THEOREM</u> For all $n \in \mathbb{N}$

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1}$$

<u>**Proof**</u> (incorrect!)

Let $\varphi(n)$ be the statement
"$1 + 2 + \cdots + 2^n = 2^{n+1}$"

Then

$\boxed{\text{Suppose } \varphi(n) \text{ is true}}$

$$1 + 2 + \cdots + 2^n = 2^{n+1}$$

So $1 + 2 + \cdots + 2^n + 2^{n+1} = 2^{n+1} + 2^{n+1}$
$= 2 \times 2^{n+1}$

So $1 + 2 + \cdots 2^n + 2^{n+1} = 2^{n+2}$

$\varphi(n+1)$

$\boxed{\text{So } \varphi(n+1) \text{ is true}}$

By induction $\varphi(n)$ is true for ~~all n~~.

<u>WRONG</u> $\varphi(1)$ says $1 + 2^1 = 2^2$
or $3 = 4$ which is false

THEOREM (CORRECT !)

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

for all $n \in \mathbb{N}$.

PROOF Let $\varphi(n)$ be the statement

"$1+2+4+\cdots+2^n = 2^{n+1}-1$"

Then

① Base case

$(n=1)$ $\quad 1 + 2^1 = 3 = 2^2 - 1$ ✓

② Induction step

Assume $n \in \mathbb{N}$ and $\varphi(n)$ is true

$$1+2+4+\cdots+2^n = 2^{n+1}-1$$

$$1+2+4+\cdots+2^n+2^{n+1} = 2^{n+1}+2^{n+1}-1$$

$$1+2+4+\cdots 2^{n+1} = 2^{n+2}-1$$

So $\varphi(n+1)$ holds.

③ hence by ① and ② and mathematical induction

$\varphi(n)$ holds for each $n \in \mathbb{N}$

as required.

THEOREM For all $n \in \mathbb{N}$

$$1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

PROOF: Let $\varphi(n)$ be the statement

"$1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$"

① Base case $\varphi(1)$ says

$$1^2 = \frac{1}{6}\, 1 \times (1+1) \times (2+1)$$

which is true

② Induction step Assume $n \in \mathbb{N}$ and that $\varphi(n)$ is true. Then

$$1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

hence

$$1^2+\cdots+n^2+(n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

$$= \frac{1}{6}(n+1)\left\{ n(2n+1) + 6(n+1) \right\}$$

$$= \frac{1}{6}(n+1)\left\{ 2n^2+7n+6 \right\}$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

$$= \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

Hence $\varphi(n+1)$ holds. So $\varphi(n) \Rightarrow \varphi(n+1)$

By ① and ② and Mathematical Induction, $\varphi(n)$ holds for all n.

THEOREM $n^2 \geqslant 2n$ for $n \geqslant 2$

PROOF: Let $\varphi(n)$ be the statement

$$n^2 \geqslant 2n$$

Base Case $(n=2)$

$$2^2 = 4 \geqslant 2 \times 2 = 4$$

So $\varphi(2)$ is true.

Induction step Assume $n \geqslant 2$ and that $\varphi(n)$ holds. Then

$$n^2 \geqslant 2n$$

So $(n+1)^2 = n^2 + 2n + 1 \geqslant 4n + 1$ $\circledast$

and $n \geqslant 2$ so

$$4n+1 \geqslant 2n+2 = 2(n+1) \quad \oplus$$

By $\circledast$ and $\oplus$

$$(n+1)^2 \geqslant 2(n+1)$$

Hence $\varphi(n+1)$ is true

Therefore $\varphi(2)$ holds &

for all $n \geqslant 2$ $(\varphi(n) \Rightarrow \varphi(n+1))$

So $\varphi(n)$ holds for all $n \geqslant 2$ in $\mathbb{N}$ by induction.

Theorem In a country with 3p and 7p coins only any integer amount over 12p can be made up

Proof: Let $\varphi(n)$ be the statement "there are $r, s \in \{0, 1, 2, \dots\}$ such that $n = 3r + 7s$"

Base Case ($n = 12$)

$$12 = 3 \times 4 + 7 \times 0$$

So $\varphi(12)$ holds.

Induction Step Suppose $n \in \mathbb{N}$, $n \geq 12$ and $\varphi(n)$ holds

Then: $n = 3r + 7s$ for some r, s

We must have $r \geq 2$ OR $s \geq 2$ (otherwise $n \leq 3 + 7 = 10$)

CASE 1 $n = 3r + 7s$; $r \geq 2$

Then $n+1 = 3(r-2) + 7(s+1)$

CASE 2 $n = 3r + 7s$; $s \geq 2$

Then $n+1 = 3(r+5) + 7(s-2)$

Either way $\varphi(n+1)$ holds

So $n \geq 12$ & $\varphi(n) \Rightarrow \varphi(n+1)$

So <u>$\varphi(n)$ holds for all $n \geq 12$</u>

<u>by mathematical induction</u>